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LETTER TO THE EDITOR

The critical dimension for lattice animals

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Abstract. Recent field theoretical calculations for lattice animals by Lubensky and Isaacson yield $d_c = 8$ as the critical dimension and provide a first-order ϵ-expansion for the exponent θ. Support for these predictions may be obtained by extending our previous work on the exact enumeration of site and bond animals on a d-dimensional simple hypercubic lattice to arbitrary d.

Recently, a field theory of branched polymers in the dilute limit has been presented by Lubensky and Isaacson (1979). Their results for branched polymers in a good solvent also apply to the statistics of lattice animals, which are important in the theory of percolation (Stauffer 1979). Assume asymptotic forms of the usual kind

$$N_b \sim b^{-\theta}\lambda_b^b, \qquad N_s \sim s^{-\theta}\lambda_s^s, \tag{1}$$

for the total number of animals with b bonds or s sites. The growth parameters λ_b and λ_s are lattice-dependent, and increase with increasing coordination number. Numerical evidence (Sykes and Glen 1976, Sykes *et al* 1976, Gaunt *et al* 1976, Gaunt and Ruskin 1978) suggests that the exponent θ is the same for both bond and site animals, and for all lattices of a given dimension. In the field theory, this result is consistent with the loop fugacity being zero at the animals' fixed point (Lubensky and Isaacson 1979). Lubensky and Isaacson find that the Gaussian approximation breaks down below a critical dimension $d_c = 8$. For dimensions $d \geq 8$, mean field theory is valid and hence $\theta = \frac{5}{2}$. Alternatively, $\theta = \frac{5}{2}$ can be obtained (Gaunt *et al* 1976, Gaunt and Ruskin 1978) from the exact results of Fisher and Essam (1961) for a Cayley tree. For $d < d_c$, Lubensky and Isaacson (1979) have derived an ϵ-expansion for θ,

$$\theta = \tfrac{5}{2} - \tfrac{1}{12}\epsilon + \ldots \qquad (\epsilon = 8 - d), \tag{2}$$

to first order in ϵ. The aim in this Letter is to test the validity of $d_c = 8$ and the ϵ-expansion (2) by extending the exact enumeration work of Gaunt *et al* (1976) and Gaunt and Ruskin (1978).

For site animals, Gaunt *et al* (1976) enumerated N_s on d-dimensional simple hypercubic lattices for $d = 2$ to 7, and $s \leq 9$ for $d = 6$ and 7, $s \leq 10$ for $d = 5$, $s \leq 11$ for $d = 4$, $s \leq 13$ for $d = 3$, and $s \leq 19$ for $d = 2$. Similarly, for bond animals, Gaunt and Ruskin (1978) enumerated N_b for $b \leq 10$ for $d = 4$, 5, 6 and 7, $b \leq 11$ for $d = 3$, and $b \leq 15$ for $d = 2$. In order to test $d_c = 8$, we have extended our data to $d = 8$ and 9 for $s \leq 9$ and $b \leq 10$. To understand how this was done, consider first the site problem. One

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Table 1. Coefficients A_ξ^s for site animals.

s	$\xi=1$	2	3	4	5	6	7	8
2	1							
3	4	1						
4	32	17	1					
5	400	348	61	1				
6	6 912	8 640	2 836	214	1			
7	153 664	254 800	129 288	21 225	758	1		
8	4 194 304	8 749 056	6 160 640	1 688 424	154 741	2 723	1	
9	136 048 896	343 901 376	313 921 008	125 055 400	20 762 073	1 123 143	9 908	1

Table 2. Coefficients α_ξ^b for bond animals.

b	$\xi=0$	1	2	3	4	5	6	7	8	9
1	1									
2	4	1								
3	32	20	1							
4	400	420	86	1						
5	6 912	10 368	4 164	370	1					
6	153 664	301 840	186 552	38 205	1 626	1				
7	4 194 304	10 223 616	8 637 760	2 934 560	343 380	7 310	1			
8	136 048 896	396 809 280	427 708 848	207 353 960	43 517 697	3 086 049	33 464	1		
9	5 120 000 000	17 408 000 000	22 888 035 968	14 551 923 200	4 552 863 040	628 406 112	27 948 467	155 444	1	
10	219 503 494 144	853 070 397 696	1 326 024 805 120	1 048 268 558 064	442 224 105 756	95 175 488 385	8 975 840 816	255 716 421	730 532	1

may write for all dimensions

$$N_1(d)=1,$$

$$N_s(d)=\sum_{\xi=1}^{s-1} A_\xi^s \binom{d}{s-\xi} \qquad (s\geq 2), \tag{3}$$

where the coefficients A_1^s, A_2^s and A_3^s are given explicitly as functions of s by Gaunt *et al* (see (2.4)). Knowledge of these three functions, together with the exact enumeration data mentioned above, is sufficient to calculate successive A_ξ^s numerically for all $s \leq 9$. These are presented in table 1. A similar procedure may be followed for the bond problem. The analogue of (3) is

$$N_b(d)=\sum_{\xi=0}^{b-1} \alpha_\xi^b \binom{d}{b-\xi} \qquad (b\geq 1), \tag{4}$$

and general expressions for α_0^b, α_1^b and α_2^b are given by Gaunt and Ruskin (see (2.4)). Numerical values of α_ξ^b for all $b \leq 10$ are given in table 2. The coefficients in tables 1 and 2, together with equations (3) and (4), enable one to calculate $N_s(d)$ for $s \leq 9$ and $N_b(d)$ for $b \leq 10$ *for arbitrary dimension.* We give the explicit values for $d = 8$ and 9 in table 3, since these are the numbers we study numerically.

The data in table 3 have been analysed by following exactly the same procedure, based upon ratio and Padé approximant techniques (Gaunt and Guttmann 1974), as was used for $d \leq 7$ (Gaunt *et al* 1976, Gaunt and Ruskin 1978). Estimates of λ and θ, and values of $\lambda^{(\sigma)}$, for both site and bond animals are presented in table 4 for $2 \leq d \leq 9$.

Table 3. Total numbers of site and bond animals per lattice site for simple hypercubic lattices of dimensions $d = 8$ and 9.

	Site animals		Bond animals	
	$d=8$	$d=9$	$d=8$	$d=9$
N_1	1	1	8	9
N_2	8	9	120	153
N_3	120	153	2 360	3 417
N_4	2 276	3 309	53 936	88 785
N_5	49 204	81 837	1 356 384	2 540 385
N_6	1 156 688	2 205 489	36 449 288	77 712 933
N_7	28 831 384	63 113 061	1 028 383 408	2 496 998 097
N_8	750 455 268	1 887 993 993	30 118 187 174	83 307 378 987
N_9	20 196 669 078	58 441 956 579	908 484 362 016	2 863 316 024 021
N_{10}			28 066 925 011 960	100 816 360 575 435

The estimates, $\lambda^{(\sigma)}$, of λ are obtained by truncating the appropriate $1/\sigma$-expansion, where $\sigma = 2d-1$, after the last term (see Gaunt and Ruskin, (3.7) and (3.8)). The results for $d = 7$ (site problem only), 8 and 9 are new; the rest are taken from Gaunt *et al* (1976) and Gaunt and Ruskin (1978), and are repeated here in order that the overall behaviour may more easily be discerned.

It can be seen from table 4 that the estimates of θ for site animals are in broad agreement with the corresponding estimates for bond animals, but have larger uncertainties. Accordingly, we focus our discussion on bond animals, since it is for these that our evidence is most compelling. In figure 1, the estimates of θ are plotted

Table 4. Summary of estimates of critical parameters for site and bond animals for simple hypercubic lattices of dimensions $d = 2$ to 9.

	Site animals			Bond animals		
d	λ	$\lambda^{(\sigma)}$	θ	λ	$\lambda^{(\sigma)}$	θ
2	4·06 ± 0·02	1·875	1·00 ± 0·05	5·210 ± 0·006	5·250	1·00 ± 0·01
3	8·35 ± 0·04	7·568	1·50 ± 0·09	10·62 ± 0·08	11·230	1·55 ± 0·05
4	13·35 ± 0·2	13·148	1·90 ± 0·15	16·3 ± 0·4	16·931	1·90 ± 0·07
5	18·8 ± 0·4	18·673	2·25 ± 0·30	22·1 ± 0·8	22·522	2·2 ± 0·1
6	24·4 ± 0·9	24·169	2·5 ± 0·4	27·75 ± 1·0	28·060	2·3 ± 0·2
7	29·5 ± 1·5	29·648	2·3 ± 0·3	33·25 ± 1·5	33·567	2·4 ± 0·2
8	35·0 ± 1·8	35·116	2·4 ± 0·3	39·0 ± 2·0	39·057	2·5 ± 0·2
9	40·5 ± 2·2	40·578	2·45 ± 0·3	44·5 ± 2·7	44·534	2·6 ± 0·35

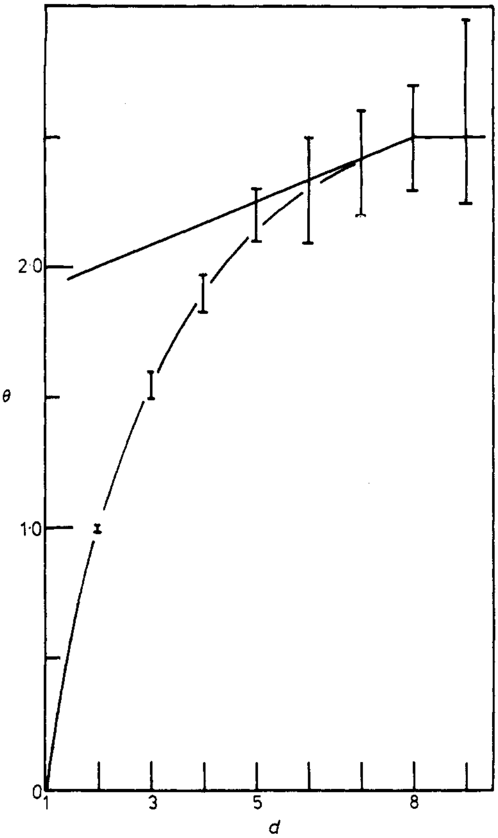

Figure 1. Estimates of θ from bond animals plotted against lattice dimensionality d. The broken curve is a smooth interpolation; the full lines show field theory predictions.

against d. It is easy to draw a smooth curve (shown broken) through all the estimates for $d \leq 8$ and which passes through the point $\theta = 0$, $d = 1$. (The exact value $\theta = 0$ for $d = 1$ follows trivially from the result $N_b(d = 1) = 1$ for all b.) The field theory predictions are drawn as full lines in figure 1; namely, the first-order ϵ-expansion result (2) for $d \leq 8$, and $\theta = \frac{5}{2}$ for $d \geqslant 8$. It should be noted that the uncertainties in θ are sufficiently large to just admit $d_c = 6$, which is the critical dimension for percolation processes. However, the results clearly favour $d_c = 8$ (over $d_c = 6$), and we think this figure provides quite strong support for the field theoretical calculations of Lubensky and Isaacson (1979). We understand that this conclusion is further supported by some recent work of Harris and Walker (1980).

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